One way of constructing the Itô integral

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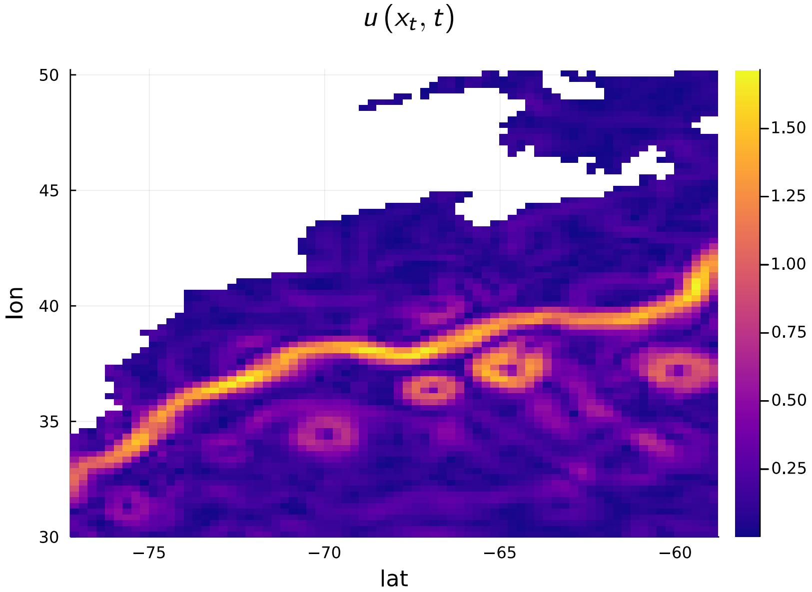
stonks

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = u(x_t, t)$$


$u\left(x_t, t\right)$
50
45
40
35
30
lon
−75
−70
−65
−60
lat
1.50
1.25
1.00
0.75
0.50
0.25

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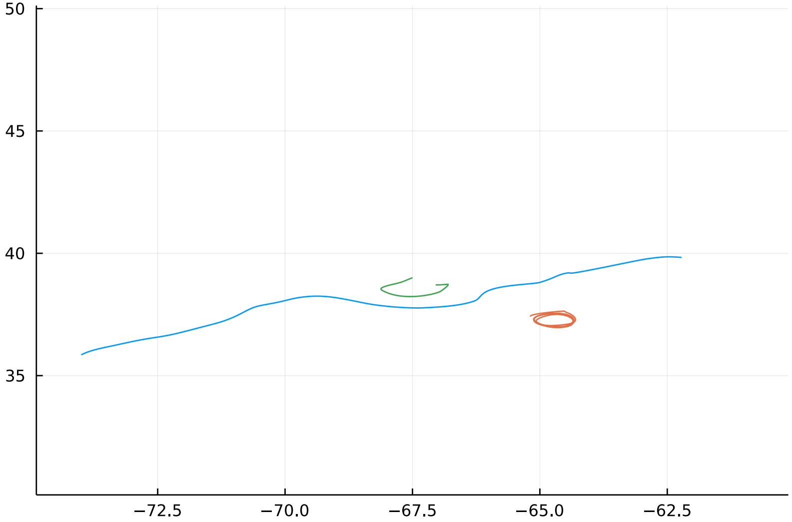

But we don't know u exactly...

Introduce some "noise" ξ_t

$$\frac{\mathrm{d}y_t}{\mathrm{d}t} = u(y_t, t) + \xi_t$$

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$$\frac{\mathrm{d}y_t}{\mathrm{d}t} = u(y_t, t) + \sigma(y_t, t)\,\xi_t$$

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But no such stochastic process ξ_t exists!

Let's discretise our ODE

$$x_{t_{k+1}} - x_{t_k} = u\left(x_{t_k}, t\right)\delta t + \sigma\left(x_{t_k}, t\right)\xi_{t_k}\delta t$$

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$$x_{t_{k+1}} - x_{t_k} = u\left(x_{t_k}, t\right)\delta t + \sigma\left(x_{t_k}, t\right)\left(W_{t_{k+1}} - W_{t_k}\right)$$

where W_t is some new stochastic process, such that

$$\xi_{t_k} = \frac{W_{t_{k+1}} - W_{t_k}}{\delta t}.$$

Set $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $s < t$.

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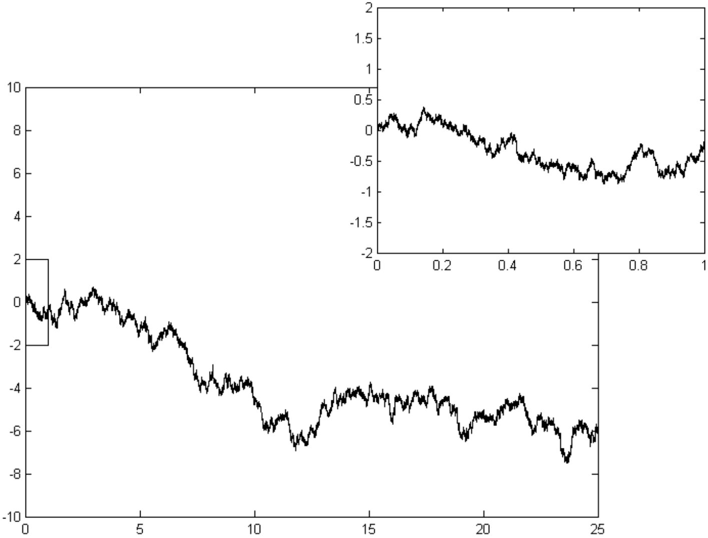

$$x_{t_k} = x_{t_0} + \sum_{k=0}^{n-1} u\left(x_{t_k}, t_k\right) \delta t + \sum_{k=0}^{n-1} \sigma\left(x_{t_k}, t_k\right) \left(W_{t_{k+1}} - W_{t_k}\right)$$

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Take $\delta t \to 0$,

$$x_t = x_0 + \int_0^t u\left(x_\tau, \tau\right) \mathrm{d}\tau + \text{“} \int_0^t \sigma\left(x_\tau, \tau\right) \mathrm{d}W_\tau \text{”}$$

$$\int_0^t \sigma\left(x_\tau, \tau\right) \mathrm{d}W_\tau \stackrel{?}{=} \lim_{\delta t \to 0} \sum_{k=0}^{n-1} \sigma\left(x_{t_k}, t_k\right)\left(W_{t_{k+1}} - W_{t_k}\right)$$

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This limit does not exist in the pointwise sense.

However, we can show that this makes sense as a limit in probability.

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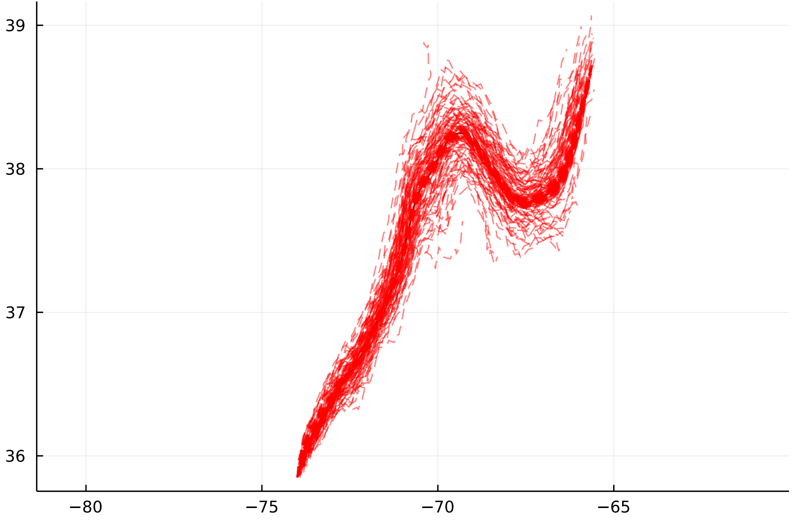
The **Itô integral**,

$$\int_0^t \sigma\left(x_\tau, \tau\right) \mathrm{d}W_\tau = \operatorname*{plim}_{\delta t \to 0} \sum_{k=0}^{n-1} \sigma\left(x_{t_k}, t_k\right)\left(W_{t_{k+1}} - W_{t_k}\right)$$

$$y_t = y_0 + \int_0^t u\left(y_\tau, \tau\right) \mathrm{d}\tau + \int_0^t \sigma\left(y_\tau, \tau\right) \mathrm{d}W_\tau .$$

$$y_t = y_0 + \int_0^t u(y_\tau, \tau)\,\mathrm{d}\tau + \int_0^t \sigma(y_\tau, \tau)\,\mathrm{d}W_\tau.$$



Further Reading

Sanjeeva Balasuriya.

Stochastic Sensitivity: A Computable Lagrangian Uncertainty Measure for Unsteady Flows.

SIAM Review, 62:781–816, Nov 2020.

Gopinath Kallianpur and P. Sundar.

Stochastic Analysis and Diffusion Processes.

Number 24 in Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, United Kingdom, first edition edition, 2014.

Bernt Øksendal.

Stochastic Differential Equations.

Universitext. Springer Berlin Heidelberg, Berlin, Heidelberg, sixth edition, 2003.