The Fokker-Planck Equation

Liam Blake

The University of Adelaide

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where $\boldsymbol{u}\colon \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is the drift, $\boldsymbol{L}\colon \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n\times m}$ is the diffusion, and $\boldsymbol{W}_t$ is m-dimensional Brownian noise.

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$$\int_0^\tau \boldsymbol{L}(\boldsymbol{y}_\tau, \tau)\,\mathrm{d}\boldsymbol{W}_t = \operatorname*{plim}_{\delta t \to 0} \sum_{[\tau_i, \tau_{i+1}] \in \mathcal{P}_{[0,t]}} \boldsymbol{L}\big(X_{\tau_i}, \tau_i\big)\left(\boldsymbol{W}\tau_{i+1} - \boldsymbol{W}_{\tau_i}\right).$$

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This can be thought of as a generalisation of a Riemann-Stieltjes integral.

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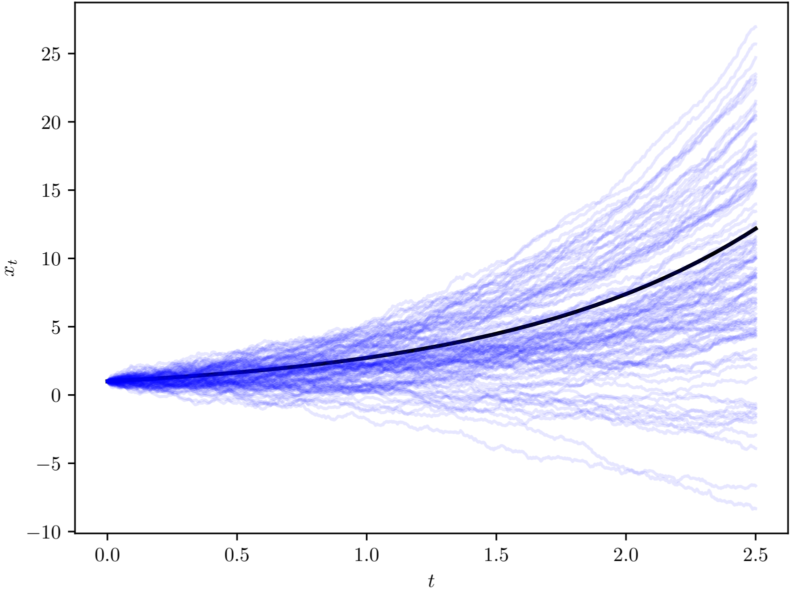
- ...but we can solve it numerically to obtain sample paths.

e.g. $\mathrm{d}X_t = X_t \, \mathrm{d}t + \mathrm{d}W_t$



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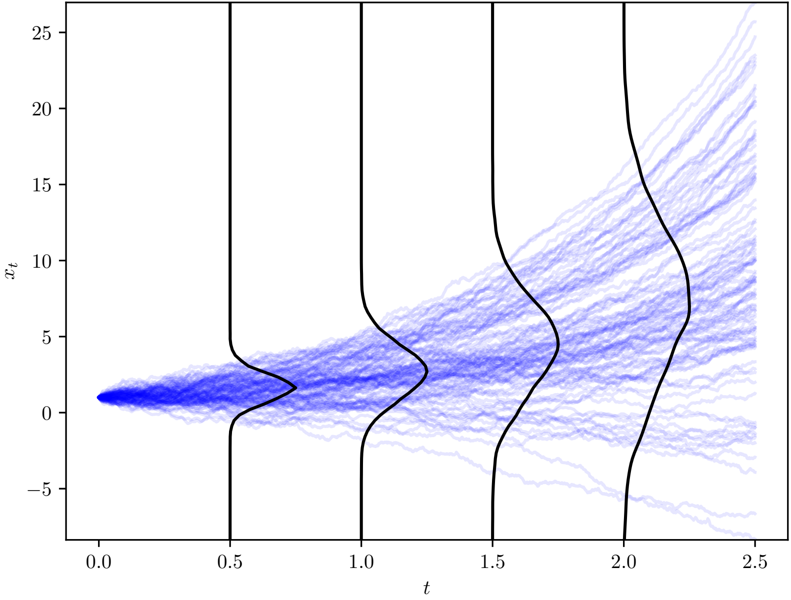

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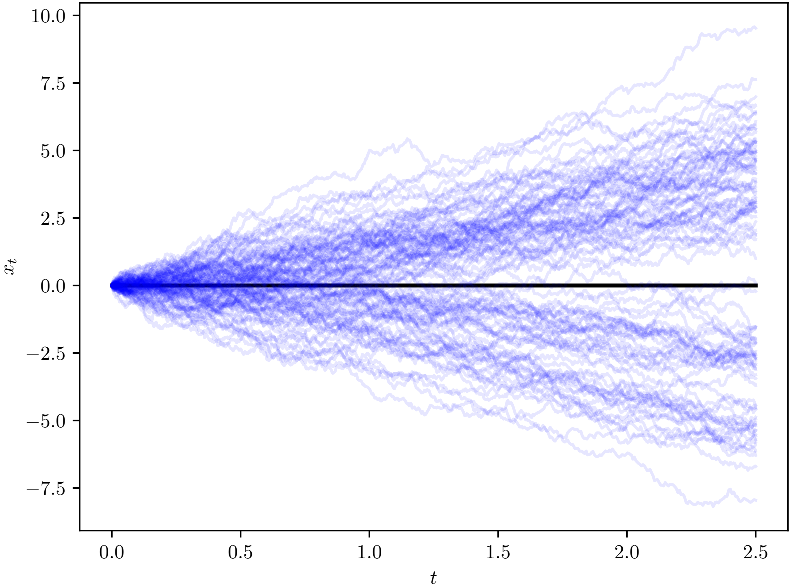

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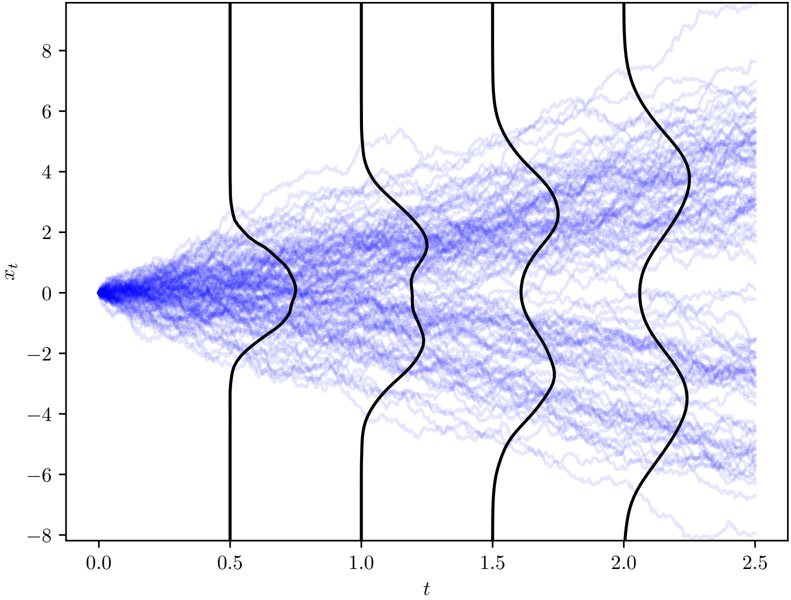

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The probability density function $\rho(\boldsymbol{x}, t)$ of $\boldsymbol{y}_t$ at time t satisfies the Fokker-Planck equation

$$\frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} = -\boldsymbol{\nabla}\cdot\big(\rho(\boldsymbol{x}, t)\,\boldsymbol{u}(\boldsymbol{x}, t)\big) + \frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot\Big(\rho(\boldsymbol{x}, t)\,\boldsymbol{L}(\boldsymbol{x}, t)\,\boldsymbol{L}(\boldsymbol{x}, t)^{\intercal}\Big)$$

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This presents an alternative to solving the SDE directly.

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- Provide a way to map uncertain initial conditions forward with SDE dynamics — this is useful in data assimilation.

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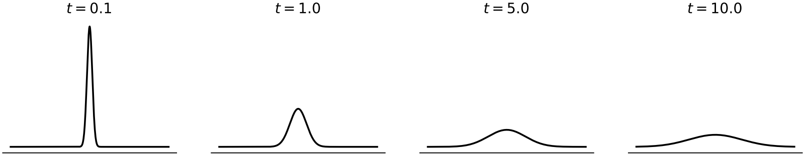
This is exactly the heat equation. The solution is $y_t \sim \mathcal{N}(0, t)$ and so

$$\rho(x,t) = \frac{1}{t\sqrt{2\pi}} \exp\left[-\frac{x^2}{2t}\right]$$



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The solution is a diffusing Gaussian with a drifting mean

$$\rho(x,t) = \frac{1}{t\sqrt{2\pi}} \exp\left[-\frac{\left(x-e^t\right)^2}{2t}\right]$$

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This means that every advection-diffusion problem can be equivalently written as a stochastic differential equation:

$$\mathrm{d}\boldsymbol{x}_t = \boldsymbol{v}(\boldsymbol{x}_t, t)\,\mathrm{d}t + D\,\mathrm{d}\boldsymbol{W}_t.$$

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One common approach is to model these as stochastic perturbations.

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$$\mathrm{d}\begin{pmatrix} \mu_t \\ \nu_t \end{pmatrix} = -\frac{1}{T_L} \begin{pmatrix} \mu_t \\ \nu_t \end{pmatrix} \mathrm{d}t + \boldsymbol{\kappa}(x_t, y_t)\, \mathrm{d}\begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix}$$

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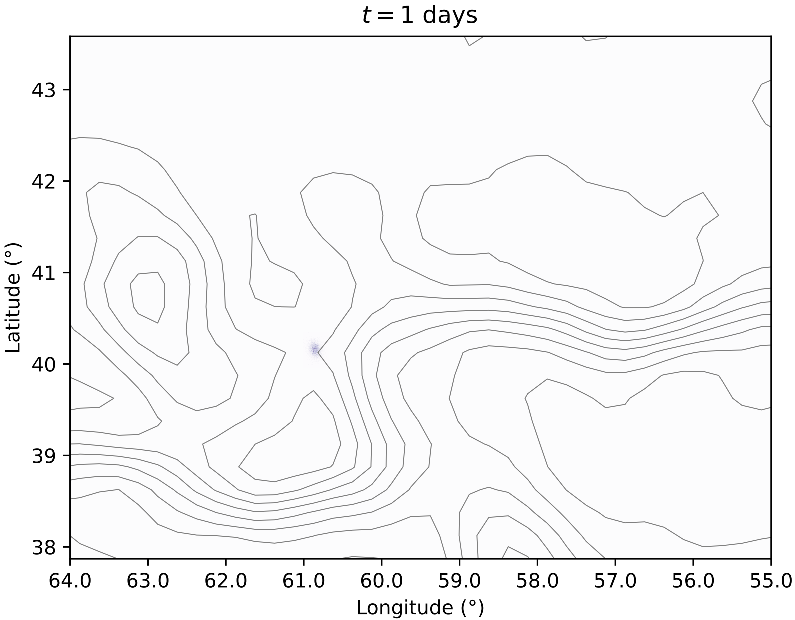
In our notation

- The state variable is $\boldsymbol{x} = (x_t, y_t, \mu_t, \nu_t)$, so $n = 4$ and $m = 2$.
- The velocity is

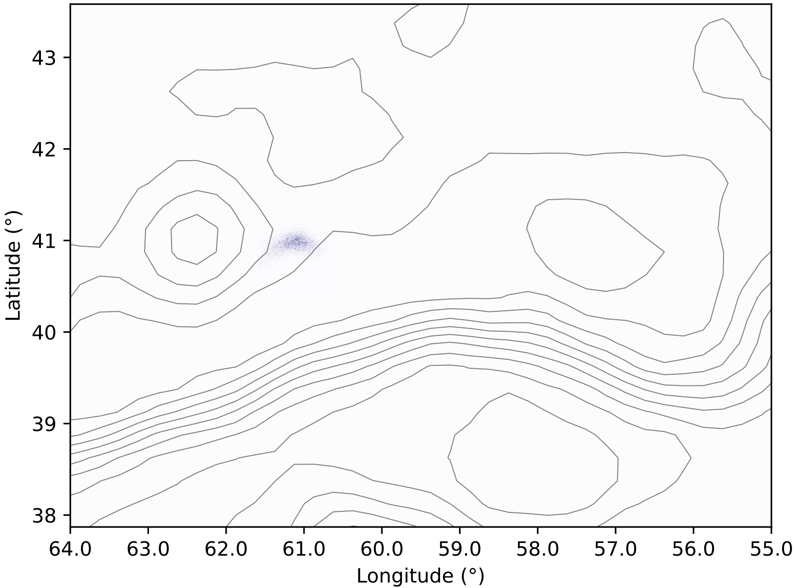
$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} u(x_t, y_t, t) \\ v(x_t, y_t, t) \\ -\frac{1}{T_L}\mu_t \\ -\frac{1}{T_L}\nu_t \end{pmatrix}$$

- The diffusion is the 4×2 matrix

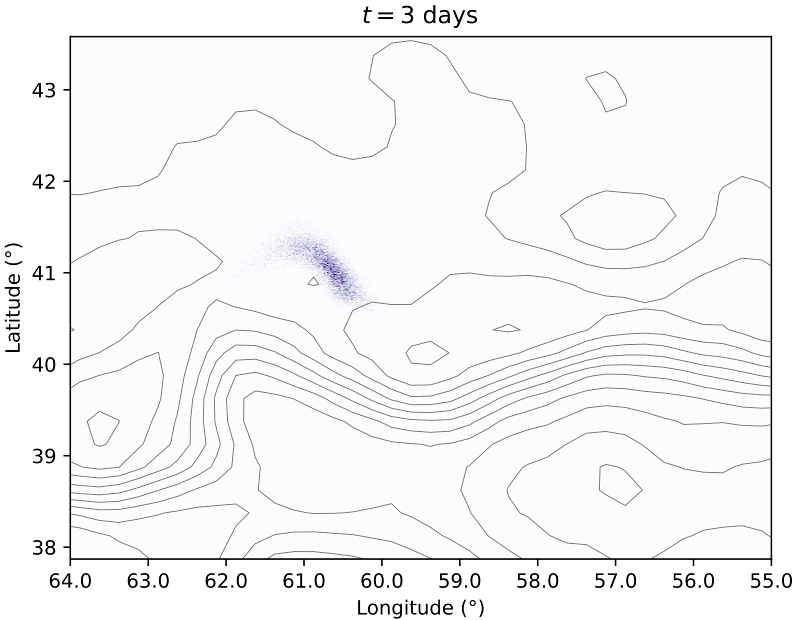
$$\boldsymbol{L}(\boldsymbol{x}, t) = \begin{pmatrix} \boldsymbol{O}_{2\times 2} \\ \boldsymbol{\kappa}(x_t, y_t) \end{pmatrix}$$

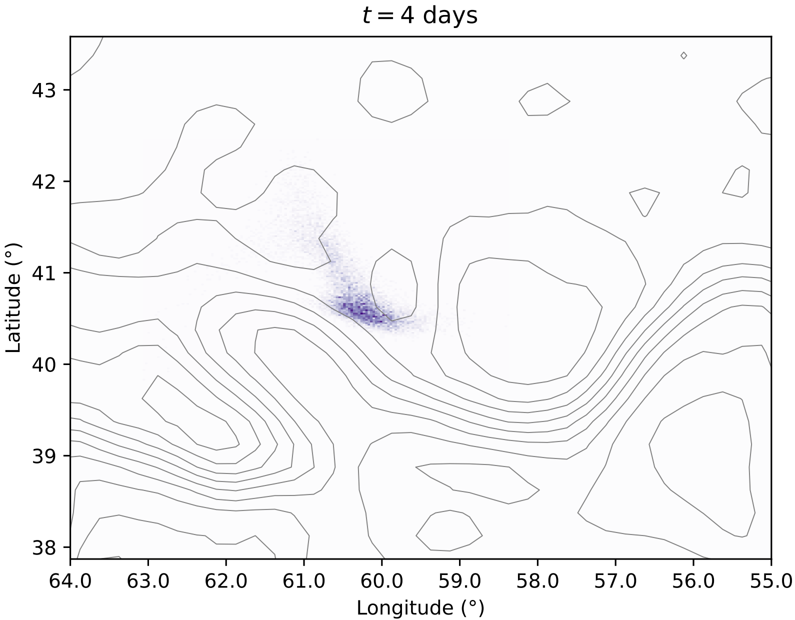

$t = 1$ days
Latitude (°)
43
42
41
40
39
38
Longitude (°)
64.0
63.0
62.0
61.0
60.0
59.0
58.0
57.0
56.0
55.0

$t = 2$ days

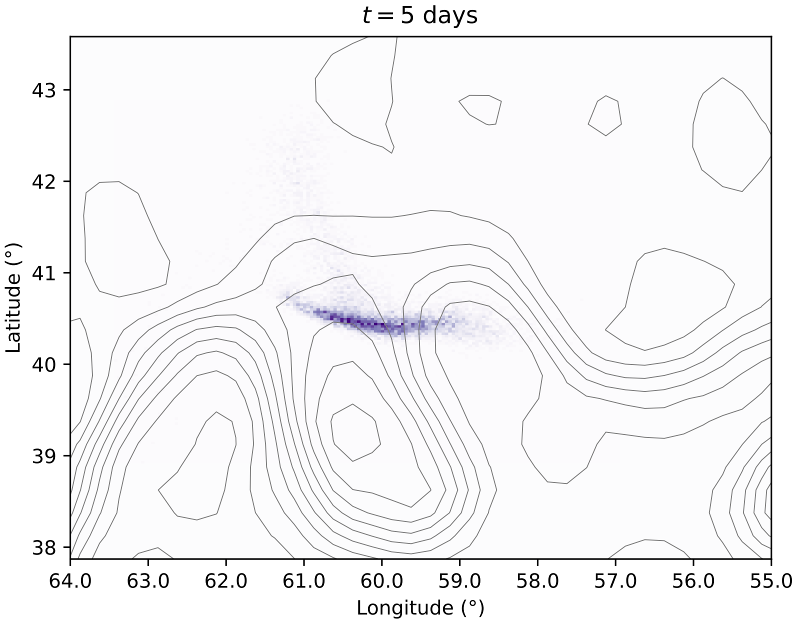


43
42
41
40
39
38
Latitude (°)
64.0
63.0
62.0
61.0
60.0
59.0
58.0
57.0
56.0
55.0
Longitude (°)


t = 3 days
Latitude (°)
Longitude (°)
43
42
41
40
39
38
64.0
63.0
62.0
61.0
60.0
59.0
58.0
57.0
56.0
55.0



$t = 4$ days
Latitude (°)
Longitude (°)
43
42
41
40
39
38
64.0
63.0
62.0
61.0
60.0
59.0
58.0
57.0
56.0
55.0



t = 5 days
Latitude (°)
43
42
41
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39
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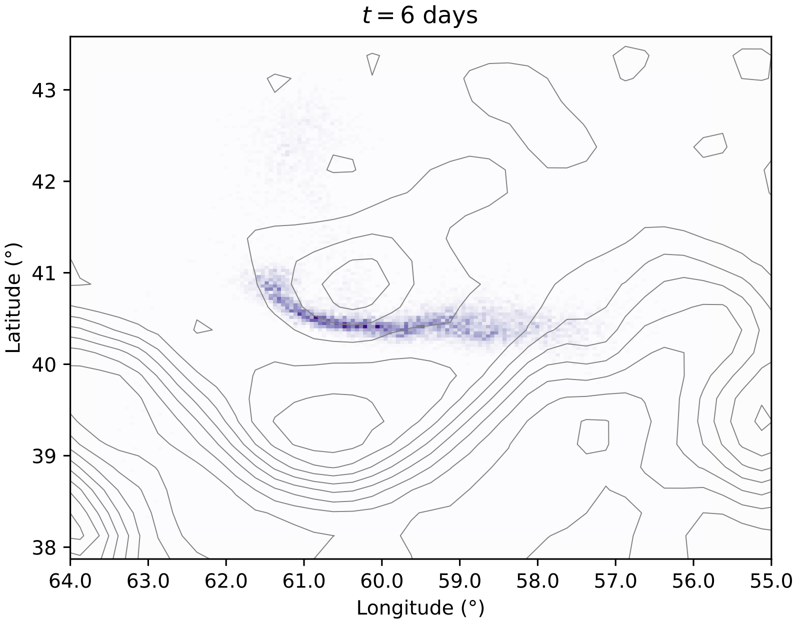
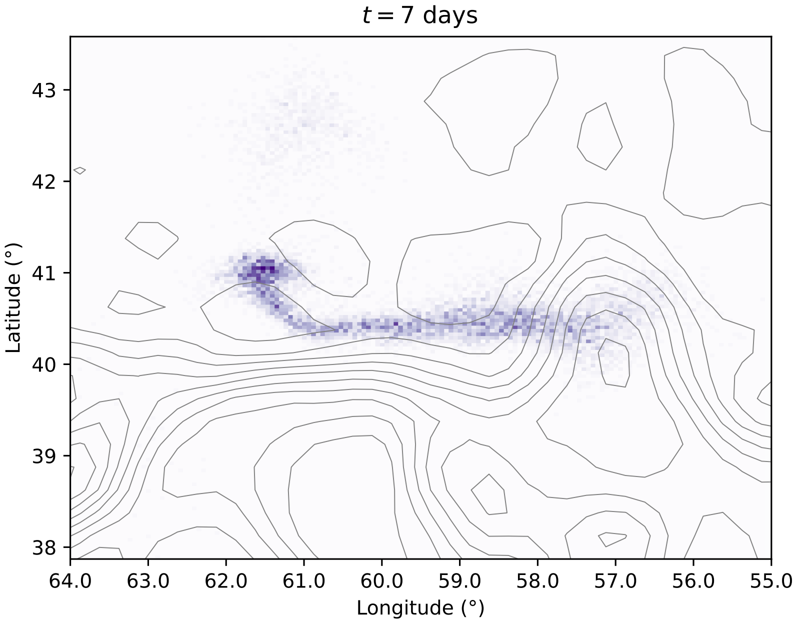
62.0
61.0
60.0
59.0
58.0
57.0
56.0
55.0
Longitude (°)



$t = 6$ days
Latitude (°)
Longitude (°)
43
42
41
40
39
38
64.0
63.0
62.0
61.0
60.0
59.0
58.0
57.0
56.0
55.0



$t = 7$ days
Latitude (°)
Longitude (°)
43
42
41
40
39
38
64.0
63.0
62.0
61.0
60.0
59.0
58.0
57.0
56.0
55.0